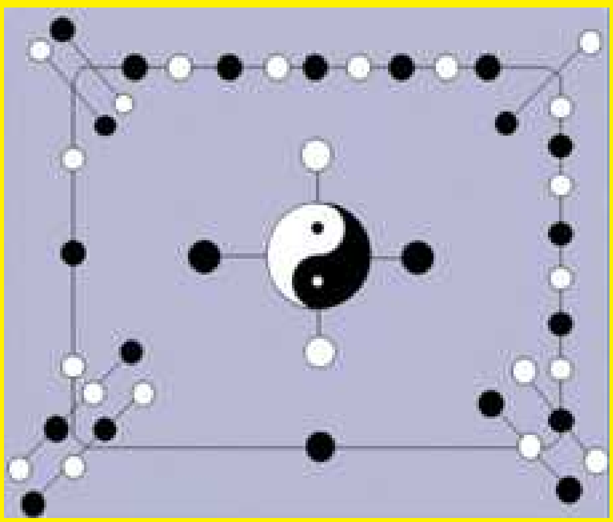
ISSN 1937 - 1055

Special Issue 1, 2018

INTERNATIONAL JOURNAL OF
MATHEMATICAL COMBINATORICS

Proceedings of the

International Conference on Discrete Mathematics and its Applications



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

November, 2018

International J.Math. Combin. Special Issue 1 (2018), 132-136

The Connectivity Number of an Arithmetic Graph

Mary Jenitha L and Sujitha S

(Department of Mathematics, Holy Cross College (Autonomous), Tirunelveli, Tamil Nadu, India)

E-mail: jeni.mathematics@gmail.com, sujivenkit@gmail.com

Abstract: The arithmetic graph V_n is defined as a graph with its vertex set is the set consists of the divisors of n (excluding 1) where n is a positive integer and $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_r^{a_r}$ where $p_i's$ are distinct primes and a_i's $\geqslant 1$ and two distinct vertices a, b which are not of the same parity are adjacent in this graph if $(a, b) = p_i$, for some i, $1 \leq i \leq r$. In this paper, we study some results related to the connectivity κ of an arithmetic graph. It is also shown that, the edge connectivity κ' and the connectivity κ are equal in arithmetic graph V_n.

Key Words: Arithmetic graph, connectivity, edge connectivity.

AMS(2010): 05C12.

§1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by ν and ϵ respectively. We consider connected graphs with at least three vertices. For basic definitions and terminologies we refer to [2].

The arithmetic graph V_n is defined as a graph with its vertex set is the set consists of the divisors of n (excluding1) where n is a positive integer and $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_r^{a_r}$ where $p_i's$ are distinct primes and $a_i's \geqslant 1$ and two distinct vertices a,b which are not of the same parity are adjacent in this graph if $(a, b) = p_i$ for some i, $1 \leq i \leq r$. The vertices a and b are said to be of the same parity if both a and b are the powers of the same prime, for instance $a = p^2$, $b = p^5$. The construction of an arithmetic graph with a given integer was introduced and studied by Vasumathi and Vangipuram in [4]. The domination parameters of an arithmetic graph were further studied by various authors in [3].

Connectivity is one of the basic concepts of graph theory. It is closely related to the theory of network flow problems. In an undirected graph G, two vertices u and v are called *connected* if G contains a path from u to v. Otherwise, they are called *disconnected*. A graph is said to be connected if every pair of vertices in the graph is connected. The *degree* of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg_G(v)$ or $d(v)$. A vertex of degree zero in G is called an *isolated vertex* and a vertex of degree one is called a *pendent vertex* or an *end-vertex* of G. The maximum and minimum degree of a graph G

[1]Proceedings of the *International Conference on Discrete Mathematics and its Applications*, Manonmaniam Sundaranar University, January 18-20, 2018.

[2]Received February 24, 2018, Accepted March 16, 2018, Edited by R. Kala.

is denoted by $\Delta(G)$ and $\delta(G)$ respectively. A *cut-vertex* (*cut-edge*) of a graph G is a vertex (edge) whose removal increases the number of components. A *vertex cut*, or *separating set* of a connected graph G is a set of vertices whose removal renders G disconnected. The *connectivity* or *vertex connectivity* $\kappa(G)$ is the number of vertices of a minimal vertex cut. A graph is called *k-connected* or *k-vertex-connected* if its vertex connectivity is k or greater. Any graph G is said to be *k-connected* if it contains at least k +1 vertices, but does not contain a set of $k-1$ vertices whose removal disconnects the graph and $\kappa(G)$ is defined as the largest k such that G is k-connected. An edge cut of G is a set of edges whose removal renders the graph G disconnected. The *edge-connectivity* $\kappa^{'}(G)$ is the number of edges of a minimal edge cut. A graph is said to be *maximally connected* if its connectivity equals its minimum degree. A graph is said to be *maximally edge-connected* if its edge-connectivity equals its minimum degree. For vertices u and v in a connected graph G, the distance $d\ (u,\ v)$ is the length of a shortest $u\ v$ path in G. Two vertices u and v of G are *antipodal* if $d(u,v) =$ *diam* G or $d(G)$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. The following theorems are used in sequel.

Theorem 1.1 ([2]) *For a connected graph G, $\kappa(G) \leqslant \kappa^{'}(G) \leqslant \delta(G)$.*

Theorem 1.2 ([2]) *A connected graph is a tree if and only if every edge is a cut edge.*

Theorem 1.3 ([2]) *A vertex v of a tree G is a cut vertex of G if and only if d(v) >1.*

Theorem 1.4 ([1]) it The number of vertices of an arithmetic graph $G = V_n$, $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_r^{a_r}$ where $p_i's$ are distinct primes, are $[(a_1+1)(a_2+1)\cdots(a_r+1)]-1$.

§2. Main Results

Theorem 2.1 *For an arithmetic graph $G = V_n$, $n = p_1^{a_1} p_2^{a_2}$ where p_1 and p_2 are distinct primes and $a_i = 1$ for all $i = 1, 2$; then the connectivity and the edge connectivity numbers are equal to 1.*

Proof Consider the arithmetic graph $G = V_n$, where n is the product of two distinct primes. The vertex set of V_n contains three vertices namely $p_1, p_2, p_1 \times p_2$.Clearly the arithmetic graph V_n is a tree containing two end vertices and an internal vertex. By theorem 1.3, the end vertices p_1 and p_2 are not cut vertices. It is clear that the internal vertex $p_1 \times\ p_2$ is the only cut vertex of V_n. Hence connectivity number $\kappa(V_n) = 1$. Also by theorem 1.2, every edge of V_n is a cut edge and hence the edge connectivity number $\kappa^{'}(V_n) = 1$. □

Theorem 2.2 *For an arithmetic graph $G = V_n$, $n = p_1^{a_1} p_2^{a_2}$ where p_1 and p_2 are distinct primes, then*

$$\kappa(V_n) = \kappa^{'}(V_n) = \begin{cases} 1 & for\ a_i = 1\ \&\ a_j > 1; i, j = 1, 2, \\ 2 & for\ a_i > 1; i = 1, 2. \end{cases}$$

Proof Consider the arithmetic graph $G = V_n$, where n is the product of two distinct primes.

Case 1. $a_i = 1$ and $a_j > 1; i, j = 1, 2$.

The vertex set of V_n is $V(V_n) = \{p_1, p_1^2, p_1^3, \cdots, p_1^{a_j}, p_2, p_1 \times p_2, p_1^2 \times p_2, \cdots, p_1^{a_j} \times p_2\}$. Clearly p_1 and p_2 are adjacent to the vertices $p_1 \times p_2, p_1^2 \times p_2, ..., p_1^{a_j} \times p_2$, so that $d(p_1) > 1$ and $d(p_2) > 1$. The vertices $p_1^2, p_1^3, ..., p_1^{a_j}$ are non adjacent to each other and are adjacent to exactly one vertex $p_1 \times p_2$ since otherwise it contradicts the definition of an arithmetic graph. Therefore $d(p_1^2) = d(p_1^3) = \cdots = d(p_1^{a_j}) = 1$. Since the graph has no isolated vertices, the minimum degree of the graph is one. Hence by theorem 1.1, $\kappa(G) \leq \kappa^{'}(G) \leq \delta(G) = 1$. Hence it is clear that, $\kappa(V_n) = 1 = \kappa^{'}(V_n)$.

Case 2. $a_i > 1; i = 1, 2$.

In this case the vertex set of V_n is $V(V_n) = \{\{p_1, p_2, p_1^2, p_2^2, p_1^3, \cdots, p_1^{a_1}, p_2^{a_2}, p_1 \times p_2, p_1^2 \times p_2, \cdots, p_1^{a_1} \times p_2, p_1 \times p_2^2, p_1 \times p_2^3, \cdots, p_1^{a_1} \times p_2^{a_2}\}$. By the definition of an arithmetic graph, p_1 and p_2 are adjacent to the vertices $p_1 \times p_2$, $p_1^2 \times p_2, \cdots, p_1^{a_1} \times p_2, p_1 \times p_2^2, p_1 \times p_2^3, \cdots, p_1^{a_1} \times p_2^{a_2}$. Therefore we have $d(p_1) > 1$ and $d(p_2) > 1$. Since the arithmetic graph is free from isolated vertices, $d(v) > 0$ for all $v \in V(V_n)$. The vertices $p_1^2, p_1^3, \cdots, p_1^{a_1}$ which are in the product of themselves with many times (till the maximum power) are adjacent to at least the vertices $p_1 \times p_2$, $p_1 \times p_2^2$,$\cdots$,$p_1 \times p_2^{a_2}$ and the vertices $p_2^2, p_2^3, \cdots, p_2^{a_2}$ are adjacent to at least the vertices $p_1 \times p_2$,$p_1^2 \times p_2, \cdots, p_1^{a_1} \times p_2$ hence its degrees are greater than one. Also, the vertices which are in the combination of two distinct primes have at least the vertices p_1 and p_2 are adjacent. Therefore $\delta(V_n) \geq 2$. But the vertex $p_1^{a_1} \times p_2^{a_2}$ is adjacent to exactly the two vertices p_1 and p_2, since otherwise it contradicts definition. Hence we find that $\delta(V_n) = 2$. By theorem 1.1, $\kappa(G) \leq \kappa^{'}(G) \leq \delta(G) = 2$. Let $S = \{p_1, p_2\}$ be the set of vertices which are adjacent to $p_1^{a_1} \times p_2^{a_2}$. Clearly the deletion of S from V_n, isolates the vertex $p_1^{a_1} \times p_2^{a_2}$. Hence $\kappa(V_n) = 2$. Also, by theorem 1.1, it is clear that $\kappa^{'}(V_n) \leqslant \delta(V_n) = 2$. Since $d(p_1^{a_1} \times p_2^{a_2}) = 2$, the removal of two edges incident at this vertex disconnects the graph. Hence $\kappa^{'}(V_n) = 2$. □

Theorem 2.3 *For an arithmetic graph $G = V_n$, $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ where p_i, $i = 1, 2, \cdots, r$ $(r > 2)$ are distinct primes and $a_i = 1$ for all $i = 1, 2, \cdots, r$ then $\kappa(V_n) = \kappa^{'}(V_n) = r$.*

Proof Consider the arithmetic graph $G = V_n$, where n is the product of more than two distinct primes and $a_i^{'}$s are equal to 1. By result 1.4, the arithmetic graph V_n contains $2^r - 1$ vertices. Among the $2^r - 1$ vertices, the vertex $p_1 \times p_2 \times \cdots \times p_r$ is adjacent to exactly r vertices namely $p_1, p_2, \cdots, p_r$. Therefore $d(p_1 \times p_2 \times \cdots \times p_r) = r$. Suppose it is adjacent to more than r vertices. Then there exists a vertex $v_i \neq p_i$, which is adjacent to $p_1 \times p_2 \times \cdots \times p_r$ and hence $(p_1 \times p_2 \times \cdots \times p_r, v_i) \neq p_i$ which contradicts the definition of an arithmetic graph. So $d(p_1 \times p_2 \times \cdots \times p_r) = r$. Also we can easily seen that the minimum degree $\delta(V_n) = r$. By theorem 1.1, it is observe that $\kappa(V_n) \leqslant r$. To prove $\kappa(V_n) = r$. Suppose $\kappa(V_n) < r$. Let $S = \{p_1, p_2, \cdots, p_{r-1}\}$ be the vertex cut of V_n such that $|S| \leqslant r - 1$. If S is deleted from V_n then it is easily seen that the vertex p_r is adjacent to at least the vertex $p_1 \times p_2 \times \cdots \times p_r$ and the vertex $p_i \times p_j$ is adjacent to either $p_1 \times p_2 \times \cdots \times p_{i-1} \times p_j \times p_{j+1} \times \cdots \times p_r$ or $p_1 \times p_2 \times \cdots \times p_i \times p_{j-1} \times p_{j+1} \times \cdots \times p_r$ and the vertex $p_i \times p_j \times p_k$ is adjacent to either $p_1 \times p_2 \times \cdots \times p_{i-1} \times p_{j-1} \times p_k \times \cdots \times p_r$ or $p_1 \times p_2 \times ... \times p_i \times p_{j-1} \times p_{k-1} \times \cdots \times p_r$ or

$p_1 \times p_2 \times \cdots \times p_{i-1} \times p_j \times p_{k-1} \times \cdots \times p_r$ and so on. This implies that the induced graph $\langle V_n - S\rangle$ is connected. Therefore we need at least r vertices to disconnect the graph. But the deletion of $S \cup \{p_r\}$, the graph is disconnected. Hence $\kappa(V_n) = r$.

Also, by Theorem 1.1 it is clear that $\kappa^{'}(V_n) \leqslant \delta(V_n)$. Since $\delta(V_n) = r$, we have $\kappa^{'}(V_n) \leqslant r$. Since $d(p_1 \times p_2 \times \cdots \times p_r) = r$, the removal of r edges incident at the vertex $p_1 \times p_2 \times \cdots p_r$, the graph V_n is disconnected and it is clear that the edge cut of V_n contains exactly r edges namely $p_1 \times p_2 \times \cdots \times p_r p_1$, $p_1 \times p_2 \times \cdots \times p_r p_2$, $\cdots p_1 \times p_2 \times \cdots \times p_r p_r$. Therefore $\kappa^{'}(V_n) = r$ and hence $\kappa(V_n) = \kappa^{'}(V_n) = r$. □

Theorem 2.4 *For an arithmetic graph $G = V_n$, $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ where $p_1, p_2, \cdots, p_r$ are distinct primes and $a_i^{'}s \geq 1$ for all $i = 1, 2, 3, \cdots, r$ and $p_i > 2$ then $\kappa(V_n) = \kappa^{'}(V_n) = r$.*

Proof We prove the theorem by considering the following four cases.

Case 1. All the $a_i^{'}s, i = 1, 2, 3, \cdots r$ are equal to one.

In this case we follow Theorem 2.3 and arrived the result.

Case 2. Some of the $a_i^{'}s$ are equal to one and the others are greater than 1.

Consider the vertex set of V_n as $V(V_n) = \{p_1, p_2, \cdots, p_r, p_1 \times p_2, \cdots, p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}\}$. Let the last vertex be $p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$ say v_1, where $a_i^{'}s$ are the maximum powers of the given distinct primes. By the definition of an arithmetic graph, we see that the only vertices which are adjacent to v_1 are $p_1, p_2, \cdots, p_r$. Hence $d(v_1) = r$. Also the minimum degree of V_n occurs at the vertex v_1. That is, $\delta(V_n) = r = d(v_1)$. By theorem 1.1, $\kappa(V_n) = \kappa^{'}(V_n) \leqslant \delta(V_n) = r$. But the removal of r vertices adjacent to v_1 makes the graph disconnected. Hence we obtained the result $\kappa(V_n) = r$. The edge connectivity $\kappa^{'}(V_n) = r$ is same as Theorem 2.3.

Case 3. All the $a_i^{'}s$ are equal and greater than 1.

Here also consider the last vertex of $V(V_n)$, say $p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$where the $a_i^{'}s$ are the maximum power of given distinct primes. By the definition of an arithmetic graph, it is clear that $p_1, p_2, \cdots, p_r$ are the only vertices which are adjacent to the vertex $p_1^{a_1} \times p_2^{a_2} \times \ldots . \times p_r^{a_r}$. The remaining proof is similar to Case 2

Case 4. All the $a_i^{'}s$ are distinct and greater than one.

Consider the last vertex in the vertex set of V_n, say $p_1^{a_1} \times p_2^{a_2} \times \cdots . \times p_r^{a_r}$ where the $a_i^{'}s$ are the maximum power of the given distinct primes. By the definition of an arithmetic graph, this vertex is adjacent to exactly r vertices namely $p_1, p_2, p_3, \cdots, p_r$. Suppose it is adjacent to any other vertex except p_i then it contradicts the definition of an arithmetic graph. The remaining proof is similar to Case 2. □

Corollary 2.5 For an arithmetic graph $G = V_n$, $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ where $p_1, p_2, \cdots, p_r$ are distinct primes and $a_i^{'}s \geq 1$ for all $i = 1, 2, 3, \cdots, r$ the connectivity number and the edge connectivity number are equal.

Proof It is obvious from Theorems 2.1, 2.2, 2.3 and 2.4. □

Remark 2.6 The arithmetic graph V_n is a maximally connected graph.

Example 2.7 Consider the arithmetic graph $G = V_{60}$. since $60 = 2^2 \times 3 \times 5$. The vertex set of G is $V(V_{60}) = \{2, 3, 5, 2^2, 2 \times 3, 2 \times 5, 3 \times 5, 2^2 \times 3, 2^2 \times 5, 2 \times 3 \times 5, 2^2 \times 3 \times 5\}$. Clearly the vertex cut and edge cut of G is $S = \{2, 3, 5\}$ and $S^1 = \{2^2 \times 3 \times 52, 2^2 \times 3 \times 53, 2^2 \times 3 \times 55\}$ respectively. Hence $\kappa(G) = \kappa^{'}(G) = \delta(G) = 3$.

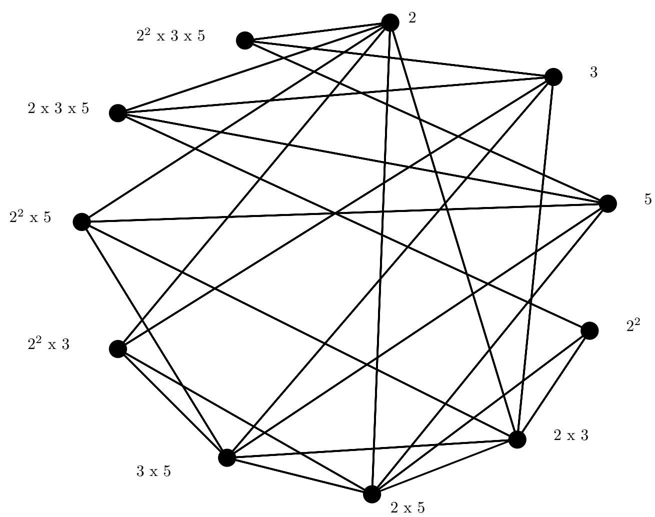


Figure 1 Arithmetic graph $G = V_{60}$

Conclusion

From the above theorems, it is clear that the connectivity number, the edge connectivity number and the minimum degree of the given arithmetic graph are equal. Also, if the given integer n is the product of more than two distinct primes then $\kappa(V_n)$ and $\kappa^{'}(V_n)$ depend on the number of distinct primes and they do not depend upon the powers of primes.

References

[1] Apostol, T.M., *Introduction to Analytic Number Theory*, Springer - verlag, Berlin, Heidelberg (1980).

[2] Bondy and Murty, *Graph Theory with Applications*, Macmillan (1976).

[3] Suryanarayana Rao K.V., Sreenivansan .V, The Split Domination in Arithmetic Graphs, *International Journal of Computer Applications,* Volume 29, No.3, (2011).

[4] Vasumathi. N., and Vangipuram. S., Existence of a graph with a given domination Parameter, *Proceedings of the Fourth Ramanujan Symposium on Algebra and its Applications*, University of Madras,Madras, 187-195 (1995).